Cauchy's integral theorem: a numerical perspective

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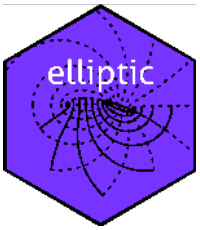
Abstract

Here I use the `myintegrate()` function of the **elliptic** package to illustrate three classical theorems from analysis: Cauchy's integral theorem, the residue theorem, and Cauchy's integral formula. In all cases, numerical values agree with analytical results to within numerical precision. Some further work is suggested.

Keywords: Elliptic functions, residue theorem, numerical integration, R.

1. Introduction



Cauchy's integral theorem and its corollaries are some of the most startling and fruitful ideas in the whole of mathematics. They place powerful constraints on analytical functions; and show that a function's local behaviour dictates its global properties. Cauchy's integral theorem may be used to prove the residue theorem and Cauchy's integral formula; these three theorems form a powerful and cohesive suite of results.

In this short document I use numerical methods to illustrate and highlight some of their consequences for complex analysis.

1.1. Cauchy's integral theorem

Augustin-Louis Cauchy proved an early version of the integral theorem in 1814; it required that the function's derivative was continuous. This assumption was removed in 1900 by Édouard Goursat at the expense of a more difficult proof; the result is sometimes known as the Cauchy-Goursat theorem and is now a cornerstone of complex analysis. Formally, in modern notation, we have:

Cauchy's integral theorem. If $f(z)$ is holomorphic in a simply connected domain $\Omega \subset \mathbb{C}$, then for any closed contour C in Ω,

$$\int_C f(z)\,dz = 0.$$

To demonstrate this theorem numerically, I will use the integration suite of functions provided with the **elliptic** package which perform complex integration of a function along a path specified either as a sequence of segments [`integrate.segments()`] or a curve [`integrate.contour()`].

Let us consider $f(z) = \exp z$, holomorphic over all of $\mathbb{C}$, and evaluate

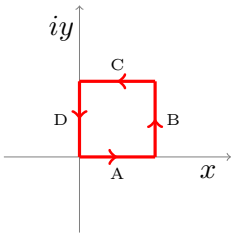


Figure 1: A square contour integral on the complex plane

$$\oint_C f(z)\,dz$$

where C is the square $0 \longrightarrow 1 \longrightarrow 1+i \longrightarrow i \longrightarrow 0$ (figure 1). Numerically:

```
> integrate.segments(exp, c(0, 1, 1+1i, 1i), close=TRUE)

[1] 1.110223e-16+0i
```

Above we see that the result is zero (to within numerical precision), in agreement with the integral theorem. It is interesting to consider each leg separately. We have

$$A = e - 1 \qquad B = e(e^i - 1) \qquad C = -e^i(e - 1) \qquad D = -(e^i - 1)$$

And taking B as an example:

```
> analytic <- exp(1)*(exp(1i)-1)
> numeric  <- integrate.segments(exp, c(1, 1+1i), close=FALSE)
> c(analytic=analytic, numeric=numeric, difference=analytic-numeric)

           analytic             numeric          difference
-1.249588+2.287355i -1.249588+2.287355i  0.000000+0.000000i
```

showing agreement to within numerical precision.

1.2. The residue theorem

residue theorem. Given U, a simply connected open subset of $\mathbb{C}$, and a finite list of points $a_1, \ldots, a_n$. Suppose $f(z)$ is holomorphic on $U_0 = U \setminus \{a_1, \ldots a_n\}$ and γ is a closed rectifiable curve in U_0. Then

$$\oint_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathrm{I}(\gamma, a_k) \cdot \mathrm{Res}(f, a_k)$$

where $\mathrm{I}(\gamma, a_k)$ is the winding number of γ about a_k and $\mathrm{Res}(f, a_k)$ is the residue of f at a_k.

The canonical, and simplest, application of this is to derive the log function by integrating $f(z) = 1/z$ along the unit circle, as per figure 2. Here the residue at the origin is 1, so the integral round the unit circle is, analytically, $2\pi i$. Numerically:

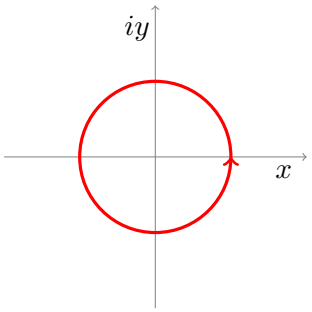


Figure 2: A circular contour integral on the complex plane

```
> u     <- function(x){exp(pi*2i*x)}
> udash <- function(x){pi*2i * exp(pi*2i*x)}
> analytic <- pi*2i
> numeric  <- integrate.contour(function(z){1/z}, u, udash)
> c(analytic=analytic, numeric=numeric, difference=analytic-numeric)

                  analytic                     numeric
0.000000e+00+6.283185e+00i -3.561641e-17+6.283185e+00i
                 difference
3.561641e-17+8.881784e-16i
```

again we see very close agreement.

1.3. Cauchy's integral formula

Cauchy's integral formula. If $f(z)$ is analytic within and on a simple closed curve C (assumed to be oriented anticlockwise) inside a simply-connected domain, and if z_0 is any point inside C, then

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z - z_0}.$$

We may use this to evaluate the Gauss hypergeometric function at a critical point. The Gauss hypergeometric function ${}_2F_1(a, b; c; z)$ is defined as

$$1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \cdots$$

Now, this series has a radius of convergence of 1 (Abramowitz and Stegun 1965); but the function is defined over the whole complex plane by analytic continuation (Buhring 1987). The **hypergeo** package (Hankin 2015) evaluates ${}_2F_1(a, b; c; z)$ for different values of z by applying a sequence of transformations to reduce $|z|$ to its minimum value; however, this process is ineffective for $z = \frac{1}{2} \pm i\sqrt{3}/2$, these points transforming to themselves. Numerically:

```
> library("hypergeo")
> z0 <- 1/2 + sqrt(3)/2i
```

```
> f <- function(z){hypergeo_powerseries(1/2, 1/3, 1/5, z)}
> f(z0)

[1] NA
```

Above we see `NA`, signifying failure to converge. However, the residue theorem may be used to evaluate ${}_2F_1$ at this point:

```
> r <- 0.1 # radius of contour
> u <- function(x){z0 + r*exp(pi * 2i * x)}
> udash <- function(x){r * pi * (0+2i) * exp(pi * 2i * x)}
> (val_residue <- integrate.contour(function(z){f(z) / (z-z0)}, u, udash) / (pi*2i))

[1] 0.7062091-0.8072539i
```

We can compare this value with that obtained by a more sophisticated [and computationally expensive] method, that of Gosper (Hankin 2015):

```
> (val_gosper <- hypergeo_gosper(1/2, 1/3, 1/5, z0))

[1] 0.7062091-0.8072539i

> abs(val_gosper - val_residue)

[1] 1.798219e-14
```

Above we see reasonable numerical agreement.

2. Conclusions

The **elliptic** package includes a suite of functionality for complex integration using numerical methods and these are used to demonstrate Cauchy's integral theorem, the residue theorem, and Cauchy's integral formula. Numerical errors are generally small.

References

Abramowitz M, Stegun IA (1965). *Handbook of Mathematical Functions*. New York: Dover.

Buhring W (1987). "An analytic continuation of the hypergeometric series." *Siam J. Math. Anal.*, **18**(3), 884–889.

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